

# Detecting quantum speedup in closed and open systems

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We construct a general measure for detecting the quantum speedup in both closed and open systems. The speed measure is based on the changing rate of the position of quantum states on a manifold with appropriate monotone Riemannian metrics. Any increase in speed is a clear signature of dynamical speedup. To clarify the mechanisms for quantum speedup, we first introduce the concept of *longitudinal* and *transverse* types of speedup: the former stems from the time evolution process itself with fixed initial conditions, while the latter is a result of adjusting initial conditions. We then apply the proposed measure to several typical closed and open quantum systems, illustrating that quantum coherence (or entanglement) and the memory effect of the environment together can become resources for longitudinally or transversely accelerating dynamical evolution under specific conditions and assumptions.

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## I. INTRODUCTION

The speed of quantum evolution determines how long it will take for a quantum state to evolve to a target state in a given process. The shortest possible time for this dynamical process, characterized by Mandelstam-Tamm [1] or Margolus-Levitin lower bound [2], is typically called the quantum speed limit (QSL) time, and this quantity plays an important and fundamental role in the study of quantum computation [3], quantum metrology [4, 5], quantum thermodynamics [6, 7], and quantum control [8, 9]. Recent decades have witnessed considerable research on the QSL time, both in closed systems [10–26] and in more general open systems [27–44]. One important discovery is the fact that entanglement is able to speed up the evolution of closed quantum systems [14, 17, 18, 22]. Further studies have shown that the memory effect of the environment can also induce dynamical acceleration in open quantum systems [31, 36]; this phenomenon has been experimentally observed in recent studies with a cavity field as the open system and the number of atoms as the controllable environment [32]. The above discoveries are of considerable interest and importance because they reveal that both entanglement and the memory effect may be beneficial for controlling dynamical processes.

However, as noted in Ref. [38], no consistent definition exists for dynamical speedup, which has been conceptualized from multiple different perspectives. For instance, in the study of the entanglement-assisted speedup of the quantum evolution of closed systems, the primary focus is on the actual evolution time  $\tau$ . Speedup occurs when the actual driving time  $\tau$  approaches the QSL time  $\tau_{QSL}$ , i.e.,  $\tau/\tau_{QSL} \rightarrow 1$  [14, 17, 18, 22]. From another point of view, we may focus on the QSL time itself, with the actual driving time  $\tau$  held fixed. In this case,  $\tau/\tau_{QSL} = 1$  implies that the evolution has already been proceeding

along the fastest possible path and that there is no capacity for further speedup, whereas for  $\tau/\tau_{QSL} > 1$ , the system is thought to possess greater capacity for potential speedup [31, 32, 36–40, 42, 43]. Therefore, there is a strong need for a well-defined measure of the speed of quantum evolution that can be employed both to detect real (not just potential) dynamical speedup and to re-examine the mechanisms of dynamical speedup in closed and open systems.

Here, we construct such a measure for detecting quantum speedup. Analogous to the instantaneous speed of an object based on the rate of change of its position in Euclidean space as calculated using the Euclidean metric, the speed of quantum evolution is defined as the changing rate of the position of quantum states on a manifold with appropriate monotone Riemannian metrics. The quantum speedup is then detected by any increase of the speed. To further clarify the mechanisms for the quantum speedup, the “speedup” is divided into two types, i.e., *longitudinal speedup* and *transverse speedup*. The former arises during the evolution process itself with fixed initial conditions, while the latter is due to a change in the initial conditions. We then apply the proposed measure to several typical closed and open quantum systems to illustrate the phenomena of longitudinal and transverse speedup. We find that quantum coherence and entanglement can serve as resources for such speedup only for specific types of quantum states in closed and open systems. In addition, the memory effect in the given examples is also confirmed to be a subtle but important factor in determining the longitudinal or transverse acceleration of the evolution process.

The paper is organized as follows: In Sec. II, we construct the measure for detecting quantum speedup based on appropriate monotone Riemannian metrics. The mechanism for longitudinal and transverse quantum speedup in closed and open systems is analyzed in Sec. III with two typical and pedagogical examples. Finally, conclusions are drawn in Sec. IV.

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## II. DETECTION OF THE SPEEDUP OF DYNAMICAL EVOLUTION

To construct the measure of dynamical speed, let us first recall several notations for quantum states. The physical state of a quantum system is represented by the density operator  $\rho$ , and the set of density operators  $\mathcal{M}_n$  that we consider here is a subset of bounded operators  $B(\mathcal{H})$  defined on the finite  $n$ -dimensional Hilbert space  $\mathcal{H}$ . Given the initial state  $\rho_0$ , the dynamics is governed by  $\rho_t = \Lambda_t \rho_0$ , where  $\Lambda_t$  is the dynamical map [45]. From the perspective of the differential geometry of quantum states, the evolved density operators constitute a manifold  $\mathcal{M}_n$ , and the dynamical map  $\Lambda_t$  can be treated as a curve on  $\mathcal{M}_n$  with parameter  $t$  [46]. We note that at every time point  $t$ , the quantum state possesses its own tangent vector space  $T$  on  $\mathcal{M}_n$ , which constitutes a tangent vector field  $\mathcal{F}_T$  on the manifold  $\mathcal{M}_n$  when the entire process of quantum evolution is considered. To obtain the length of the curve, an appropriate monotone Riemannian metric tensor field  $g : \mathcal{F}_T \times \mathcal{F}_T \rightarrow F$  (where  $F$  denotes the scalar field on  $\mathcal{M}_n$ ) should be given. Then, the line element of the curve is defined as [Note 1]

$$ds^2 := (dl)^2 \text{ with } dl := \|\dot{\rho}_t\| dt, \quad (1)$$

where  $\|X\| := \sqrt{g(X, X)}$  denotes the length of any tangent vector  $X \in T$  and  $\dot{\rho}_t := \partial_t \rho_t \in \mathcal{F}_T$  [46, 47]. The length of the curve is then expressed as

$$l := \int dl = \int_0^t \sqrt{g(\partial_{t'} \rho_{t'}, \partial_{t'} \rho_{t'})} dt' \quad (2)$$

The above construction allows us to define the instantaneous speed of quantum evolution as

$$S := \dot{l} = \sqrt{g(\dot{\rho}_t, \dot{\rho}_t)}. \quad (3)$$

To derive an explicit form of the above quantity, we must select an appropriate monotone Riemannian metric. If the quantum state is pure, then the Fubini-Study metric is the natural choice. However, if the quantum system exists in mixed states, then there are infinitely many monotone Riemannian metrics [46]. According to the theorem of Morozova, Chentsov, and Petz, any monotone Riemannian metric can be represented in the unified form  $g(X, Y) := \frac{1}{4} \langle X, K^{-1}(Y) \rangle$ , with  $X, Y \in T$ . Here  $\langle \cdot, \cdot \rangle$  denotes the Hilbert-Schmidt inner product, and  $K$  is a positive superoperator defined as  $K := f(L_\rho R_\rho^{-1}) R_\rho$ , where  $R_\rho(X) := X\rho$  and  $L_\rho(X) := \rho X$ , and  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an operator monotone function  $f(x) = xf(x^{-1})$  being normalized by  $f(1) = 1$  [48]. Therefore, the speed of quantum evolution is given by

$$S = \frac{1}{2} \sqrt{\langle \dot{\rho}_t, K^{-1}(\dot{\rho}_t) \rangle}. \quad (4)$$

To obtain the above measure in an explicit form, we may rewrite the density operator in the form of its spectral

decomposition  $\rho_t = \sum_k p_k |\Phi_k\rangle \langle \Phi_k|$ , with  $0 < p_k < 1$  and  $\sum_k p_k = 1$ . In the Morozova-Chentsov-Petz formalism [46], Eq. (4) can be immediately written as

$$S = \frac{1}{2} \sqrt{\sum_{k,l} c(p_k, p_l) |\langle \Phi_k | \dot{\rho}_t | \Phi_l \rangle|^2}, \quad (5)$$

where the symmetric function  $c(x, y) = 1/[yf(x/y)]$  is the so-called Morozova-Chentsov function [48] related to our chosen Riemannian metrics.

We note that any monotone Riemannian metric can be employed to evaluate the instantaneous speed of quantum evolution with Eq. (5) as long as the quantum state is in the interior of the space of quantum states, i.e.,  $0 < p_k < 1$ . However, because a measure of speed should be applicable to both closed and open systems, it is hoped that Eq. (5) could be continuously extended to the boundary of the manifold, i.e., one or more of  $p_k$  vanishes. As can be seen from Eq. (5), two kinds of singularity, i.e.,  $c(p_k, p_k) = 1/p_k \rightarrow \infty$  and  $c(p_k, p_l) \rightarrow \infty$  ( $k \neq l$ ), may appear when we are trying to make an extension to the boundary. Fortunately, the first singularity can be removed by employing a strategy of replacing  $p_k$  with  $q_k$  ( $q_k^2 = p_k$ ) [46]. However, to our knowledge, there exists only a few monotone Riemannian metrics being able to avoid the second kind of singularity [Note2], such as the symmetric logarithmic derivative metric, with  $c_{SL}(x, y) = \frac{2}{x+y}$ , and the Wigner-Yanase metric, with  $c_{WY}(x, y) = \frac{4}{(\sqrt{x} + \sqrt{y})^2}$ . The above consideration is justified when we take the pure state  $\rho_t = |\psi\rangle \langle \psi|$  as an example, whose line element is the well known  $ds_{FS}^2 = \langle d\psi_\perp | d\psi_\perp \rangle$  of the Fubini-Study metric [27, 46]. Here  $|d\psi_\perp\rangle := |d\psi\rangle - \langle \psi | d\psi \rangle |\psi\rangle$  is the component of  $|d\psi\rangle$  orthogonal to  $|\psi\rangle$ . It is convenient to check that, for pure states, the line elements of the symmetric logarithmic derivative metric and the Wigner-Yanase metric will reduce to  $ds_{SL}^2 = \langle d\psi_\perp | d\psi_\perp \rangle = ds_{FS}^2$  and  $ds_{WY}^2 = 2\langle d\psi_\perp | d\psi_\perp \rangle = 2ds_{FS}^2$ , respectively.

In terms of above strategy and selected metrics, we have

$$S = \sqrt{\sum_k \dot{q}_k^2 + \sum_{k \neq l} c_{SL(WY)}(p_k, p_l) \frac{p_k(p_k - p_l)}{2} |\langle \Phi_l | \dot{\Phi}_k \rangle|^2}, \quad (6)$$

where we have utilized  $c(x, y) = c(y, x)$  and  $\langle \Phi_l | \dot{\Phi}_k \rangle + \langle \dot{\Phi}_l | \Phi_k \rangle = 0$ . As a special case when the system is a closed system with a pure state  $\rho_t = |\psi\rangle \langle \psi|$ , it is simple to check that Eq. (6) reduces to

$$S = \frac{\epsilon |\langle d\psi_\perp | \dot{\psi} \rangle|}{\sqrt{\langle d\psi_\perp | d\psi_\perp \rangle}}, \quad (7)$$

with  $\epsilon = 1, \sqrt{2}$  for the symmetric logarithmic derivative metric and the Wigner-Yanase metric, respectively. In the following, we will focus on the symmetric logarithmic

derivative metric as an example; extension to the Wigner-Yanase metric and other potential appropriate metrics [49] is straightforward.

Equation (6) can now serve as a basis for detecting quantum speedup. Clearly, any increase in speed is a signature of dynamical speedup:

$$\partial_\xi S > 0, \quad (8)$$

where  $\xi$  is the concerned dynamical parameter. In the following, Eq. (8) (or  $\partial_\xi S$ ) will be employed as a detection (or a measure) for quantum speedup. To clarify the mechanisms underlying speedup phenomena, it is of necessity to distinguish two types of “speedup”: (i) *longitudinal speedup* (i.e.,  $\xi = t$ ), which arises during the evolution process itself with certain initial states and fixed environmental parameters, and (ii) *transverse speedup* (i.e.,  $\xi \in \{\xi_1, \xi_2, \xi_3, \dots\}$  and  $\partial_t \xi_i = 0$ ) due to a change in the initial conditions (parameterized by  $\xi_i$ ), such as the initial states and environmental parameters.

### III. EXAMPLES

In what follows, we apply the measure constructed above to several simple but pedagogical examples and illustrate the mechanisms corresponding to both longitudinal speedup and transverse speedup.

#### A. Closed systems: a physical model of spin precession

##### 1. Single-qubit case

Consider a physical model of spin precession, with a spin- $\frac{1}{2}$  system subjected to a uniform static external magnetic field in the  $z$  direction. The Hamiltonian of the system can be written as [50] ( $\hbar = 1$ )

$$H = \frac{\omega}{2} \sigma_z, \quad (9)$$

where  $\omega$  is the energy difference between the two spin eigenstates and  $\sigma_z := |1\rangle\langle 1| - |0\rangle\langle 0|$  is the usual Pauli operator. If the system is initially prepared in the superposition state  $|\psi_0\rangle = \alpha|1\rangle + \beta|0\rangle$ , with  $|\alpha|^2 + |\beta|^2 = 1$ , then the time evolution of the spin is given by  $|\psi_t\rangle = \alpha \exp(-i\omega t/2)|1\rangle + \beta \exp(i\omega t/2)|0\rangle$ . Using Eq. (7), we have

$$S = |\alpha\beta|\omega. \quad (10)$$

Therefore, transverse speedup (i.e.,  $\partial_\alpha S > 0$ ) can be achieved by enhancing the initial coherence of the spin state. With  $\alpha(\beta)$  given initially in Equation (10), the system will evolve at a uniform speed (i.e., longitudinal speedup will never occur, for  $\partial_t S \equiv 0$ ). In particular, the speed of evolution will be zero when  $\alpha(\beta) = 0$ . It is easily understood, for example, that if the system is initially in

state  $|1\rangle$ , then the evolved state will be  $\exp(-i\omega t/2)|1\rangle$ , which is precisely identical to  $|1\rangle$ . In fact, both state  $|1\rangle$  and state  $\exp(-i\omega t/2)|1\rangle$  belong to the same ray in projective Hilbert space; hence, the system has not evolved at all.

##### 2. Two-qubit case

The previous single-qubit case can be generalized to a bipartite system. Under the assumption that no interaction exists between the two spins, the evolution is governed by the following Hamiltonian:

$$H = H^A \otimes I^B + I^A \otimes H^B \quad (11)$$

with  $H^{A(B)} = \omega \sigma_z^{A(B)}/2$ , where  $I^{A(B)}$  is the identity operator of the spin  $A(B)$ . For convenience and without loss of generality, the initial state is first set to  $|\varphi_0\rangle = \alpha|11\rangle + \beta|00\rangle$ . The corresponding evolved state is then written as  $|\varphi_t\rangle = \alpha \exp(-i\omega t)|11\rangle + \beta \exp(i\omega t)|00\rangle$ . According to Eq. (7), we obtain

$$S = 2|\alpha\beta|\omega, \quad (12)$$

which implies that the longitudinal speedup (i.e.,  $\partial_t S \equiv 0$ ) will never occur, but the transverse speedup (i.e.,  $\partial_\alpha S > 0$ ) may be realized by adjusting the initial states. In fact, the speed in equation (12) is related to the entanglement of the initially prepared state. To demonstrate this fact, we employ Wootters’s concurrence  $C$  to measure the entanglement of the bipartite system [51]. Because the evolved density operator is of the “X” type, the concurrence can be easily deduced to be  $C = 2|\alpha\beta|$  (related to initial state). Hence, the evolution speed is given by

$$S = C\omega, \quad (13)$$

illustrating an interesting phenomenon in which quantum entanglement is able to induce transverse speedup (i.e.,  $\partial_C S = \omega > 0$ ) through control of the initial states.

Nevertheless, we note that not all entangled states possess such a property, e.g., if the system is initially prepared in the state  $|\phi_0\rangle = \alpha|10\rangle + \beta|01\rangle$ , it is easy to check, using Eq. (7), that  $S = 0$  because  $|\phi_t\rangle \equiv |\phi_0\rangle$ . However, the entanglement of this state is always  $C = 2|\alpha\beta|$  and is fully uncorrelated with the speed of quantum evolution. Therefore, the entanglement induced transverse speedup is fully state-dependent in this example.

#### B. Open systems: a physical model of two-level atoms coupled to leaky cavities

##### 1. Single-qubit case

We now turn to the case of an open system. Consider a two-level atom coupled to a leaky vacuum cavity with the following Hamiltonian [45]:

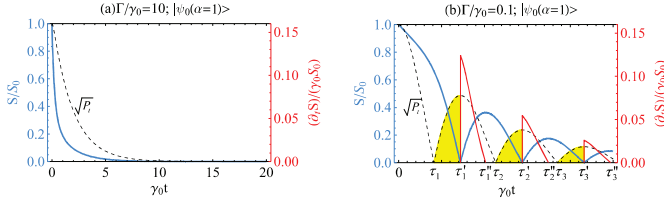


FIG. 1: Comparison among the normalized speed of quantum evolution,  $S/S_0$  (blue curve); the detection for memory effects,  $\sqrt{P_t}$  (dashed curve); and the measure for longitudinal speedup,  $(\partial_t S)/(\gamma_0 S_0)$  (red curve), as functions of  $\gamma_0 t$ , for a single-qubit open system with fixed initial state  $|\psi_0(\alpha=1)\rangle = |1\rangle$  in (a) a memoryless environment ( $\Gamma/\gamma_0 = 10$ ), and (b) a memory environment ( $\Gamma/\gamma_0 = 0.1$ ), respectively. From a phenomenological point of view, the longitudinal speedup in (b) within  $\gamma_0 t \in [\tau'_n, \tau''_n]$  is driven by the previous accumulation of memory effects (yellow-shaded regions within  $\gamma_0 t \in [\tau_n, \tau'_n]$ ).

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k a_k a_k^\dagger + i \sum_k \zeta_k (a_k^\dagger \sigma_- - a_k \sigma_+), \quad (14)$$

where  $\omega_0$  is the resonant transition frequency of the atom between the ground state  $|0\rangle$  and the excited state  $|1\rangle$ ;  $\sigma_+ := |1\rangle\langle 0|$  and  $\sigma_- := |0\rangle\langle 1|$  are the Pauli raising and lowering operators, respectively; and  $\omega_k$  and  $a_k$  ( $a_k^\dagger$ ) denote the frequency and annihilation (creation) operator, respectively, of the  $k$ th cavity mode, with  $\zeta_k$  being the corresponding coupling constant. The reduced density matrix of the atom, with an initial state  $\rho = (\rho_{mn})$  ( $m, n = 0, 1$ ), takes the form [45]

$$\rho_t = \begin{pmatrix} \rho_{11} P_t & \rho_{10} \sqrt{P_t} \\ \rho_{01} \sqrt{P_t} & 1 - \rho_{11} P_t \end{pmatrix}, \quad (15)$$

where  $P_t$  is related to the excited state population of the atom. For arbitrary initial states, the speed of quantum evolution can be acquired by Eq. (6), but in most cases, only numerical calculation is feasible. For simplicity, here, we consider the speed of evolution with an initial state of  $|\psi_0\rangle = \alpha |1\rangle + \sqrt{1-\alpha^2} |0\rangle$  ( $\alpha \in [0, 1]$ ). Then, the spectral decomposition of Eq. (15) can be obtained by  $\rho_t = \sum_{k=\mp} p_k |\Phi_k\rangle \langle \Phi_k|$  with  $p_\mp = (1 \mp \eta)/2$ ;  $|\Phi_\mp\rangle = (b_\mp |1\rangle + |0\rangle)/\sqrt{1+b_\mp^2}$ , where  $\eta = \sqrt{1-4\alpha^4 P_t + 4\alpha^4 P_t^2}$  and  $b_\mp = -(1-2\alpha^2 P_t \pm \eta)/(2\alpha\sqrt{1-\alpha^2}\sqrt{P_t})$ . According to Eq. (6), we have

$$S = \frac{\alpha |\dot{P}_t|}{2} \sqrt{\frac{1-(1-\alpha^2)P_t}{P_t(1-P_t)}}. \quad (16)$$

To be specific, let us examine a switchable environment with the Lorentzian spectral density  $J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \Gamma}{(\omega_0 - \omega)^2 + \Gamma^2}$ , where  $\gamma_0$  is the Markovian-limit decay rate and  $\Gamma$  is the spectral width [45]. Hence,  $P_t =$

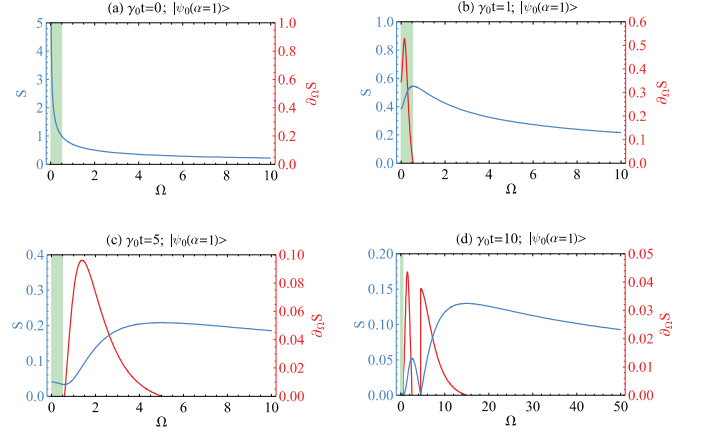


FIG. 2: Comparison between the speed of quantum evolution,  $S$  (blue curve); and the measure for transverse speedup,  $\partial_\Omega S$  (red curve), as functions of  $\Omega$  ( $=\gamma_0/\Gamma$ ), for a single-qubit open system prepared in initial state  $|\psi_0(\alpha=1)\rangle = |1\rangle$  with fixed point in time at (a)  $\gamma_0 t = 0$ ; (b)  $\gamma_0 t = 1$ ; (c)  $\gamma_0 t = 5$ ; and (d)  $\gamma_0 t = 10$ , respectively. A narrower spectral width  $\Gamma$  (i.e., larger  $\Omega$ ) corresponds to a stronger memory effect [ $\Omega \in (0, 1/2)$  for Markovian (green-shaded) region, and  $\Omega \in (1/2, \infty)$  for non-Markovian region].

$e^{-\Gamma t} [\cos(\frac{\kappa t}{2}) + \frac{\Gamma}{\kappa} \sin(\frac{\kappa t}{2})]^2$  with  $\kappa = \sqrt{2\gamma_0 \Gamma - \Gamma^2}$ , which can be controlled by the width of the spectrum:  $\Gamma/\gamma_0 > 2$  represents the memoryless (Markovian) region, and  $\Gamma/\gamma_0 < 2$  corresponds to the non-Markovian region with memory. The maximal speed occurs at  $t = 0$ , with

$$S_0 := \lim_{t \rightarrow 0} S = \alpha^2 \sqrt{\frac{\Gamma \gamma_0}{2}}. \quad (17)$$

Obviously, it can be confirmed that if  $\Gamma/\gamma_0 > 2$ , then Eq. (16) decreases monotonically to zero, implying that no longitudinal speedup occurs during the evolution [e.g., see in Fig. 1(a)]. However, if  $\Gamma/\gamma_0 < 2$ , then longitudinal speedup will occur. To illustrate this phenomenon, the normalized speed of quantum evolution,  $S/S_0$  (blue curve); and the measure for quantum speedup,  $(\partial_t S)/(\gamma_0 S_0)$  (red curve) with  $\Gamma/\gamma_0 = 0.1$  and  $\alpha = 1$  (i.e.,  $|\psi_0(\alpha=1)\rangle = |1\rangle$ ) is depicted in Fig. 1(b). The longitudinal speedup ( $\partial_t S > 0$ ) regions are within  $\gamma_0 t \in [\tau'_n, \tau''_n]$  ( $n \in \mathbb{Z}^+$ ), where  $\tau'_n/\gamma_0 = 2n\pi/\kappa$  and  $\tau''_n/\gamma_0$  denote the solutions to the transcendental equation  $\Gamma \tan(\kappa t/2) = \kappa \tanh(\Gamma t/2)$ . This phenomenon can be phenomenologically explained in terms of the memory effects of the environment [52, 53]. The memory effect of the above model is detected by  $\sqrt{P_t}$  [54], which is drawn as the dashed curve in Fig. 1(b), with the memory regions (marked in shades of yellow) corresponding to  $\gamma_0 t \in [\tau_n, \tau'_n]$  ( $n \in \mathbb{Z}^+$ ), where  $\tau_n/\gamma_0 = 2[n\pi - \arctan(\kappa/\Gamma)]/\kappa$ . From a phenomenological perspective, the longitudinal speedup is driven by the accumulation of the previous memory effect of the environment.

Meanwhile, to study the memory effect on the transverse speedup, we can select the driving time and then adjust the initial parameters of the environment. To be

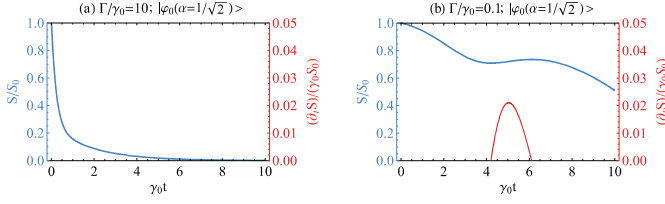


FIG. 3: Comparison between the normalized speed of quantum evolution,  $S/S_0$  (blue curve); and the measure for longitudinal speedup,  $(\partial_t S)/(\gamma_0 S_0)$  (red curve), as functions of  $\gamma_0 t$ , for a two-qubit open system initially prepared in state  $|\varphi_0(\alpha = 1/\sqrt{2})\rangle = (|11\rangle + |00\rangle)/\sqrt{2}$ ; and surrounded by (a) a memoryless environment ( $\Gamma/\gamma_0 = 10$ ), and (b) a memory environment ( $\Gamma/\gamma_0 = 0.1$ ), respectively.

clear, the speed of quantum evolution,  $S$  (blue curve); and the measure for transverse speedup,  $\partial_\Omega S$  (red curve) are plotted in Fig. 2 versus  $\Omega$  ( $=\gamma_0/\Gamma$ ), with fixed driving time. As clearly shown in Fig. 2(b), transverse speedup ( $\partial_\Omega S > 0$ ) may occur even in the Markovian region [green shaded,  $\Omega \in (0, 1/2)$ ]. When the driving time is longer [e.g., Fig. 2(c) and (d)], the transverse speedup ( $\partial_\Omega S > 0$ ) will take place in the non-Markovian region [ $\Omega \in (1/2, \infty)$ ].

## 2. Two-qubit case

Consider the case of a bipartite system of independent two-level atoms, each locally coupled to a leaky vacuum cavity. The dynamics of this open system is determined by each atom-cavity pair in the same manner illustrated in the single-qubit case. Consider, for instance and simplicity, the entangled initial state  $|\varphi_0\rangle = \alpha|11\rangle + \sqrt{1-\alpha^2}|00\rangle$  ( $\alpha \in [0, 1]$ ); the evolved state can be immediately obtained [55]:

$$\rho_t = \begin{pmatrix} \alpha^2 P_t^2 & 0 & 0 & \alpha\sqrt{1-\alpha^2}P_t \\ 0 & \alpha^2 P_t(1-P_t) & 0 & 0 \\ 0 & 0 & \alpha^2 P_t(1-P_t) & 0 \\ \alpha\sqrt{1-\alpha^2}P_t & 0 & 0 & 1-2\alpha^2 P_t + \alpha^2 P_t^2 \end{pmatrix}. \quad (18)$$

Then, the spectral decomposition of Eq. (18) is obtained by  $\rho_t = \sum_{k=1,2,\mp} p_k |\Phi_k\rangle \langle \Phi_k|$  with  $p_1 = p_2 = \alpha^2 P_t(1-P_t)$ ,  $p_\mp = (1-2\alpha^2 P_t + 2\alpha^2 P_t^2 \mp \delta)/2$ ;  $|\Phi_1\rangle = |01\rangle$ ,  $|\Phi_2\rangle = |10\rangle$ ,  $|\Phi_\mp\rangle = (d_\mp|11\rangle + |00\rangle)/\sqrt{1+d_\mp^2}$ , where  $\delta = \sqrt{1-4\alpha^2 P_t + 4\alpha^2 P_t^2}$ ,  $d_\mp = -(1-2\alpha^2 P_t \pm \delta)/(2\alpha\sqrt{1-\alpha^2}P_t)$ . According to Eq. (6), we have

$$S = \alpha |\dot{P}_t| \sqrt{\frac{1-2P_t+2P_t^2}{2P_t(1-P_t)(1-2\alpha^2 P_t + 2\alpha^2 P_t^2)}}. \quad (19)$$

It can be confirmed that the maximal instantaneous speed also occurs at  $t = 0$  with  $S_0 = \alpha\sqrt{\Gamma\gamma_0}$ . In addition, Eq. (19) also implies that longitudinal speedup ( $\partial_t S > 0$ ) occurs in a memory environment, precisely as shown in Fig. 3.

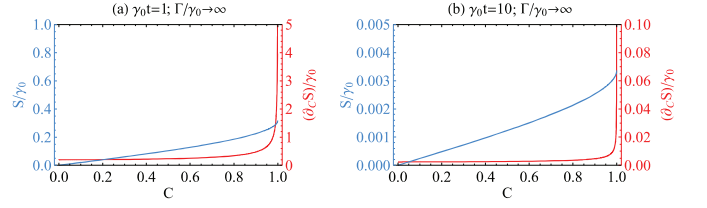


FIG. 4: Comparison between the speed of quantum evolution,  $S/\gamma_0$  (blue curve); and the measure for transverse speedup,  $(\partial_C S)/\gamma_0$  (red curve), as functions of  $C$  (entanglement of initial two-qubit state measured by Concurrence) in a Markovian-limit environment ( $\Gamma/\gamma_0 \rightarrow \infty$ ) with fixed point in time at (a)  $\gamma_0 t = 1$ ; (b)  $\gamma_0 t = 10$ , respectively.

On the other hand, to study the transverse speedup, e.g., the influence of initial entanglement  $C$  ( $=2\alpha\sqrt{1-\alpha^2}$ ) on the speedup, we may select an environment (e.g., the Markovian limit with  $\Gamma \rightarrow \infty$  and  $P_t = \exp(-\gamma_0 t)$ ), and fix the driving time  $t$ . Equation (19) thereby reduces to

$$\frac{S}{\gamma_0} = \frac{1}{2} \sqrt{\frac{(1-\sqrt{1-C^2})P_t(1-2P_t+2P_t^2)}{(1-P_t)[1-(1-\sqrt{1-C^2})P_t(1-P_t)]}}. \quad (20)$$

As is clearly shown in Fig. 4, stronger entanglement of initial state will give rise to the transverse speedup ( $\partial_C S > 0$ ).

However, we note that if the system is prepared in the initial state, e.g.,  $|\phi_0\rangle = \alpha|10\rangle + \beta|01\rangle$ , then we have  $S = \frac{|\dot{P}_t|}{2\sqrt{P_t(1-P_t)}}$  [irrelevant to  $\alpha(\beta)$ ], implying that the entanglement of initial state is fully uncorrelated with the speed in this example, i.e., the transverse speedup ( $\partial_C S > 0$ ) never occurs under such circumstances, which bears a resemblance to the case of example A-2.

## IV. CONCLUSION

We have constructed a well-defined geometrical measure for detecting the real speedup of quantum evolution. Furthermore, the quantum speedup has been subdivided into longitudinal and transverse types. The former focuses on the time evolution process with fixed initial conditions, while the latter concerns influence of the initial conditions on quantum speedup. With such constructed measure, the mechanisms for quantum speedup have been explored within several typical closed and open systems, thereby demonstrating the fact that quantum coherence/entanglement as well as the memory effect may serve as resources for longitudinally or transversely accelerating the evolution of quantum states under specific conditions in these examples.

We have analyzed the mechanisms for both longitudinal and transverse speedup with simple examples, and it will be of great interest and significance to perform further investigations on much complex physical systems,

such as controllable atoms [32], quantum dots [56], or nitrogen-vacancy centers [57] confined within microcavities.

## NOTES

Note 1: The notation  $ds^2$  in differential geometry can be understood from two different perspectives [47]. In the first, it is considered to be the metric tensor  $g$  itself expressed in the cotangent vector space. The second relates  $ds^2$  to square of the length of the tangent vector. In this work, we adopt the latter view.

Note 2: Other well known metrics such as the right logarithmic derivative metric and the Bogoliubov-Kubo-Mori metric [49] are not appropriate under the framework of our proposed measure, for they can not be continuously extended to the boundary of the manifold.

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